J. Fluid Mech. (1967), *vol.* 30, *part* 4, *pp.* 809–811
Printed in Great Britain

Dual solutions of the boundary-layer equations at a point of attachment

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(Received 20 April 1967)

This paper indicates the existence of a dual solution of the boundary-layer equations for the flow near a nodal point of attachment of the external flow; especial note is made of the non-axisymmetric boundary-layer flow when the external flow is axisymmetric.

It is well known that the flow of a viscous incompressible fluid near a stagnation point of attachment is of a simple form. If the external flow is irrotational and P is such a point on the surface of the body then the external flow near P has components $\{ax, by, -(a+b)(z-\delta)\}$ where the surface is represented by $z=0$, the tangent plane at P, and the x-axis is chosen so that $a>0$; b is any constant such that $a \geqslant b \geqslant -a$ and δ is the displacement thickness of the boundary layer. The velocity components (u,v,w) of the flow in the boundary layer are

$$u = axf'(\eta), \quad v = ayg'(\eta), \quad w = -\nu^{\frac{1}{2}}a^{\frac{1}{2}}\{f(\eta)+g(\eta)\}, \tag{1}$$

where $\eta = a^{\frac{1}{2}}z/\nu^{\frac{1}{2}}$ is the usual boundary-layer variable. The differential equations which f and g must satisfy are

$$f''' + (f+g)f'' + 1 - (f')^2 = 0, \tag{2}$$

$$g''' + (f+g)g'' + c^2 - (g')^2 = 0, \tag{3}$$

where $c \equiv b/a$ and a dash denotes differentiation with respect to η. The boundary conditions for (2), (3) are

$$\left.\begin{array}{lll} f = g = f' = g' = 0 & \text{when} & \eta = 0, \\ f' \to 1, \quad g' \to c & \text{as} & \eta \to \infty. \end{array}\right\} \tag{4}$$

Now for large values of η the general asymptotic expansions for f, g, as obtained from (2), (3) are, provided $c \neq -1$,

$$1-f' \sim A_1 e^{-\frac{1}{2}\chi^2}\chi^{-\{(3+c)/(1+c)\}}\{1+O(\chi^{-1})\} + B_1\chi^{\{2/(1+c)\}}\{1+O(\chi^{-1})\}, \tag{5}$$

$$c-g' \sim A_2 e^{-\frac{1}{2}\chi^2}\chi^{-\{(1+3c)/(1+c)\}}\{1+O(\chi^{-1})\} + B_2\chi^{\{2c/(1+c)\}}\{1+O(\chi^{-1})\}, \tag{6}$$

where A_1, A_2, B_1, B_2 are constants, $(1+c)^{\frac{1}{2}}\chi \equiv \eta(1+c)-\alpha-\beta$ and α, β are respectively the limits of $(\eta-f)$, $(c\eta-g)$ as $\eta\to\infty$ (see Davey (1961): the present g is Davey's g/c, and our equations (5) and (6) contain a correction of sign). In addition to (4) it is usual to require that the solution for f, g shall be such that the velocity in the boundary-layer flow approaches the free-stream value more rapidly than neighbouring solutions, so that even when c is negative, B_1 and B_2 must be zero.

What is not well known is that for every value of c in the range $-1 < c \leqslant 1$ there are nevertheless two distinct solutions for f, g satisfying all the required conditions. Dual solutions have been given by Libby (1967) for some negative values of c and by Davey & Schofield (1967) for the special case $c = 0$. In his paper Libby suggested that dual solutions existed only for $c < 0$, on the basis of a physical argument. This is not, however, the case. The dual solutions are continuous through $c = 0$ and we have obtained the dual profiles of f, g for several positive values of c, including $c = 1$. In figure 1 we compare values of $f_d''(0)$, $g_d''(0)$ for

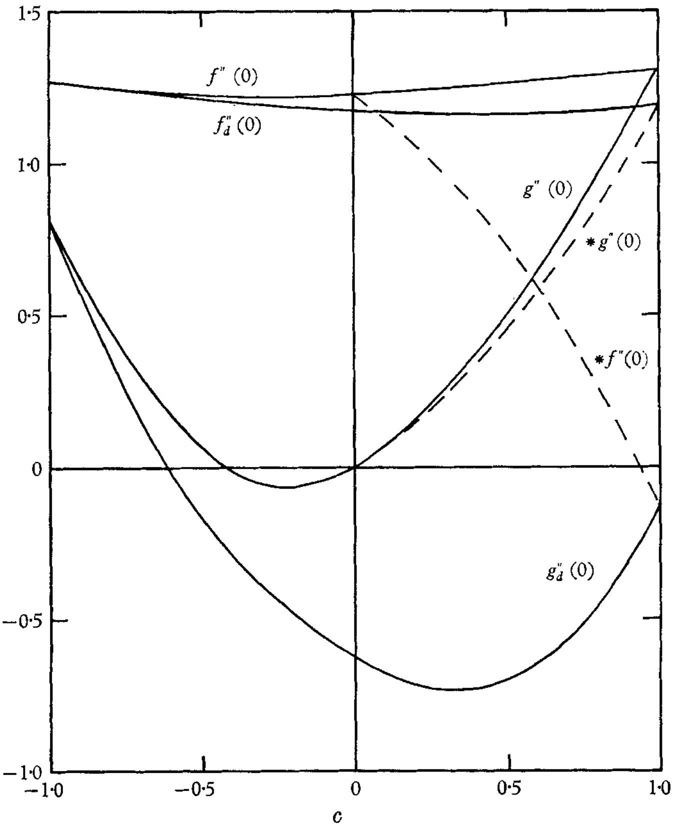

FIGURE 1. Values $f_d''(0)$, $g_d''(0)$ of the dual solutions compared to the usual values $f''(0), g''(0)$ for $1 \geqslant c \geqslant -1$. (See text for definitions of $*f$ and $*g$.)

the dual solutions with the values $f''(0)$, $g''(0)$ for the usual boundary-layer solution. The profiles for f_d are very typical and their variation with c is slight. Those for g_d all contain a region in the inner layer of reverse flow, which increases gradually with c. The case $c = 1$ is of especial interest because the external flow is axisymmetric but there is still a (locally unique) solution of (2), (3), (4) with $B_1 = B_2 = 0$ such that $f \neq g$; that is the dual flow in the boundary layer is asymmetric. For this case $f_d''(0) = 1{\cdot}1904$ and $g_d''(0) = -0{\cdot}1317$; f_d' and g_d' are shown in figure 2. The value of δ, the displacement thickness, is $4{\cdot}424$, much larger than the usual value of $0{\cdot}568$ due to blockage caused by the reversed flow near the y-axis.

The dual solutions for $c > 1$ may be used to generate further solutions of (2) and (3) in the range $0 < c < 1$ by means of the relation

$$*f(c, \eta) = c^{\frac{1}{2}} g(c^{-1}, c^{\frac{1}{2}}\eta), \tag{7}$$

$$*g(c, \eta) = c^{\frac{1}{2}} f(c^{-1}, c^{\frac{1}{2}}\eta) \tag{8}$$

(where c is positive). The regular solutions are invariant under this transformation. These further solutions, $*f$, $*g$, differ from the dual solutions as a class only in notation. They are also shown on figure 1. The limit as $c \to 0$ is non-uniform.

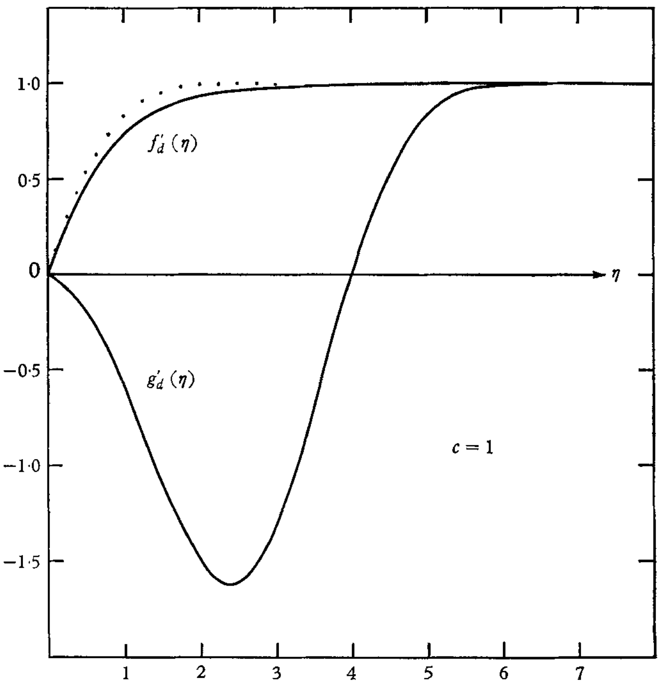

FIGURE 2. Values $f_d'(\eta)$, $g_d'(\eta)$ for $c = 1$. (Dots represent the usual $f = g$ solution.)

The dual solutions may be of mathematical interest only, but even so their very existence is important. Moreover, the preparation of this short paper was felt to be worthwhile in view of the interesting asymmetrical nature of the dual solution for $c = 1$ when the external flow is as at an axisymmetrical stagnation point. Such a solution may be thought of as a finite disturbance solution of the usual axisymmetric solution.

Our thanks go to Mr E. J. Watson for some helpful discussions and encouragement. This work was done as part of the general research programme of the National Physical Laboratory.

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